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Optimal test methods for determining material parameters

Méthodes d'essai optimales pour déterminer les paramètres du matériau

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ABSTRACT: Measuring the parameters that control the deformability and strength of soils through either laboratory experiments or *in situ* testing is critical for numerous applications in geotechnical engineering. While image- and wave-based techniques are increasingly prevalent, there is a perpetual need for techniques capable of sensing local, nonlinear properties, for which mechanical testing is the only viable option. Existing methods for inferring mechanical properties have evolved largely by trial and error, and there is no general, systematic approach for evaluating one possible approach against another. As a first step toward addressing these challenges, this paper describes a quantitative metric that can discriminate between different types of mechanical tests with respect to how well they are able to recover the true mechanical properties of the material. The metric is devised by (1) creating a min-max optimization of parameter sensitivities, considering the local and global topological properties of the forward model, and (2) evaluating the metric for fundamental material tests.

RÉSUMÉ: La mesure des paramètres qui contrôlent la déformabilité et la résistance des sols, par des expériences en laboratoire ou des testes in situ, est essentielle pour de nombreuses applications en génie géotechnique. Alors que les techniques basées sur l'image et les ondes sont de plus en plus répandues, il existe un besoin perpétuel de techniques capables de détecter des propriétés locales non linéaires, pour lesquelles les testes mécaniques sont la seule option viable. Les méthodes existantes pour déduire les propriétés mécaniques ont évolué en grande partie par essai et erreur et il n'y a pas d'approche générale et systématique pour évaluer une approche possible par rapport à une autre. Comme première étape pour relever ces défis, cet article décrit une métrique quantitative qui peut discriminer entre différents types de testes mécaniques en ce qui concerne leur capacité à récupérer les véritables propriétés mécaniques du matériau. La métrique est conçue en (1) créant une optimisation min-max des sensibilités des paramètres en tenant compte des propriétés topologiques locales et globales du modèle direct et (2) en évaluant la métrique pour les testes de matériaux fondamentaux.

KEYWORDS: sensitivity, optimization, laboratory testing, *in situ* testing, parameter identification

1 INTRODUCTION

Currently, there is no quantitative, unbiased technique to compare different testing methods for measuring the mechanical properties of materials. This holds for the plethora of tests devised for laboratory and field testing of soils as well as material testing in general. For example, for the simple choice of whether to use force or displacement control for uniaxial compression of metals, although certain standards such as ASTM E9 (2019) give ample direction to use displacement control, no guidance or mention is given for a force-controlled test. Quantitative comparison of material test methods is critical for accurate and efficient assessment of material test methods for various objectives. Key benefits are (1) the ability to specify the optimal testing method for a specific application and (2) the ability to inform the development of new testing methods for optimal performance. A test can more accurately return the true mechanical properties of a material when the sensitivity of the material parameter/s of interest are maximized (Hill 1998) while minimizing sensitivity to all other components of the test (Taguchi et al. 2000).

This paper proposes a quantitative metric, which combines the work of Hill (1998) and Taguchi et al. (2000) to introduce a min-max sensitivity analysis, allowing the user to directly compare material tests. A thought experiment involving the determination of a spring constant is first introduced in Section 2, which gives context to the implementation of the proposed quantitative metric. Then, the mathematical construction of the proposed quantitative metric is outlined in detail in Section 3. Finally, in Section 4, implementation of the quantitative metric for the fundamental material tests of spring extension and uniaxial compression provides information on the optimal test in the context of the proposed quantitative metric.

2 THOUGHT EXPERIMENT

As an explicit example, this paper first considers the simple problem of measuring spring constant k for a linear spring. Moreover, the analysis based on the proposed quantitative metric aims to answer the following question: if faced with

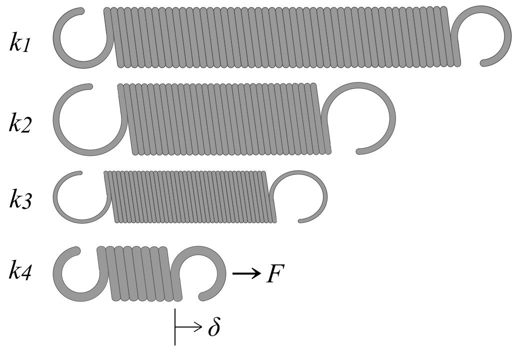

Figure 1. Springs of various spring constants (k_1, k_2, k_3, and k_4) determined by imposing/measuring force, F, and displacement, δ

several springs of unknown spring constants (Figure 1), what is the optimal test to run on these specimens? In other words, what type of test, which may be unique to each spring, will most accurately return the true spring constant?

For the simple case of determining a spring constant, one has a single choice to make: (1) apply a force to the spring and measure the displacement, or (2) prescribe a displacement and measure the resulting force. The first option, referred to as a "force-controlled" test, is perhaps the most common for evaluating spring constants. The simplicity of being able to hang weights from a spring and measure the resulting displacement is often easier than attempting to control the displacement and measure the force (e.g., using a load cell), which represents the second choice of a "displacement-controlled" test. The analysis presented in this paper does not consider the practicality of conducting either force- or displacement-controlled tests and considers only the accuracy to which the spring constant can be calculated. Assuming idealized testing conditions where deviations in force and displacement are the only sources of error, inaccuracy is introduced in one of two ways for a force-controlled test: (1) the measurement of displacement differs from the true value, or (2) the prescribed force differs from the intended value. Similarly, for a displacement-controlled test, inaccuracy arises due to errors in either (1) measured force or (2) prescribed displacement.

The quantitative metric is applied to the extension of a spring in order to make a direct comparison of force-controlled and displacement-controlled tests, as well as determine the optimal testing configuration. The same analysis is then extended to investigate the uniaxial compression test, retaining some of the simplicity of the spring test while incorporating more testing parameters.

3 CONSTRUCTION OF QUANTITATIVE METRIC

Construction of the quantitative metric is non-trivial, and the following sections detail the choices made in converging on the form proposed in this work:

$$Q = \frac{1}{n}\sum_{i=1}^{n} w_i \left(\frac{\partial \alpha}{\partial \beta_i}\frac{\beta_{i,avg}}{\alpha}\right)^2 \quad (1)$$

Variables in Eq. (1) are defined as follows: n = number of test parameters, w_i = i^{th} component of a vector of given weights, α = output parameter, β_i = i^{th} component of a vector of input parameters, $\partial\alpha/\partial\beta_i$ = i^{th} component of a vector of the sensitivities of the output parameter to the input parameters, and $\beta_{i,avg}$ = i^{th} component of a vector of the average input parameters over their expected ranges.

The general form of the quantitative metric is

$$Q = \frac{1}{m}\left[\sum_{i=1}^{n} w_i \left[\left(\frac{\partial \alpha}{\partial \beta_i}\frac{\beta_{i,norm}}{\alpha_{norm}}\right)^p\right]^{1/q}\right]^{1/r} \quad (2)$$

In this work, sensitivities $\partial\alpha/\partial\beta_i$ are considered as the main component for optimization, and their selection is further discussed in Section 3.1. By implementing weights w_i, individual sensitivities can be minimized or maximized, which is also further discussed in Section 3.1. The summation includes the overall normalization by m, as well as the exponents p, q, and r, which are all positive integers selected by the user. The selection of these integers is outlined in Section 3.2. The normalization of all the sensitivities (to produce dimensionless quantities) is achieved by including $\beta_{i,norm}/\alpha_{norm}$, as discussed in detail in Section 3.3.

3.1 *Sensitivities and min-max optimization*

Sensitivity ($\partial\alpha/\partial\beta_i$) is a measure of how much a change in the input will change the output. By investigating this as the measure of the quality of a test, the influence of all aspects of the test on the output can be assessed. Particularly relevant applications of sensitivity to the analysis presented here are works in robust design and topological sensitivity analysis (Park et al., 2006; Eschenauer et al., 1994; Novotny et al., 2003), soil and groundwater parameter determination (Hill 1992; Hill 1998; Calvello & Finno 2004), population biology (Heppell et al. 2000; Link & Doherty 2002) and other environmental modeling applications for ranking, mapping or screening of data (Pianosi et al. 2016). Topological sensitivity analysis uses sensitivities to determine redundant components of a design that can be removed. Robust design assesses areas that have minimal influence on the design, such that design or manufacturing tolerances can be increased. Studies pertaining to groundwater and soil parameter determination use sensitivities to determine which parameters have the greatest influence on the output and therefore are most easily discovered. Finally, population biology and other environmental modeling applications use the sensitivities of animal populations to various events as a means of investigating the past, present, and future of our natural world. All the previously mentioned fields of research focus on local sensitivities, which consider only one set of parameters for any given analysis. The implementation of sensitivity analysis presented in this paper requires the analysis of global sensitivities. In global sensitivity analysis, consideration must be given to the variation of parameters across their respective parameter space. The sensitivity function is used here to refer to the variation of sensitivity across the parameter space.

Positive and negative weights w_i can be applied to individual sensitivities to minimize and maximize these quantities, respectively. The sensitivity of the output to the relevant material parameters to be determined should be *maximized* to enhance test accuracy. At the same time, the sensitivity of the output to all other inputs should simultaneously be *minimized* to reduce their influence on the output. Such min-max optimization is achieved by minimizing the value of Q and including negative weights for sensitivities that should be maximized. By including negative weights, one is minimizing the negative of corresponding terms, resulting in a maximization of the term. Weights other than unity can be added to increase or decrease the significance of particular sensitivities. However, such a possibility is not addressed in this work, where weights w_i take values of −1 or +1 depending on whether the corresponding sensitivity should be maximized or minimized, respectively.

3.2 *Sensitivity combinations*

A combination technique is required for the analysis of multiple parameter sensitivities through a single quantitative metric, denoted here by Q. In this work, sensitivities are combined through simple summation. There are various methods available for which most cases result in similar or the same outcomes of the optimization problem. A selection of options is provided here, along with applicable reasoning as to the proposed method.

Normalization of Q by the number of parameter sensitivities assessed allows for comparison across tests with a varying number of parameters. Therefore, $m = n$ in Eq. (2) for all the following applications.

The metric given by Eq. (1) arises for a particular choice of exponents p, q, and r in Eq. (2). A sum of absolute values is achieved when $p = q = 2$ and $r = 1$. This method is particularly effective when large and isolated errors are present in a data set (Dielman 1986; Ge 1997), although, in this application, the forward models are known and free of error. Alternatively, an absolute value is undesirable as it introduces complications for further analytical analysis due to the non-smoothness of the resulting function. A "sum of squares" is achieved with Eq. (2) when $p = 2$ and $q = r = 1$. This method theoretically produces the same result as the sum of absolute values when no error is present and is especially easy to work with mathematically. Although it does not perform well with outliers in a data set, it has been shown to perform well with a normally distributed error (Ge 1997). The square root of the sum of squares is achieved with Eq. (2) when both $p = r = 2$ and $q = 1$. Although commonly used in similar applications (Hill 1998; Calvello & Finno 2004), this is undesirable for the specific application here. The addition of the square root hinders the possibility of using a negative metric in the presence of maximized sensitivities. It is therefore determined that the mean sum of squares is the most suitable combination technique due to the mathematical tractability and the same performance in optimization with the presence of a lack of error. Future work may consider alternatives, including the various options described above.

3.3 Sensitivity normalization

The parameter sensitivities are dimensional and require normalization to be correctly summed. Normalization must be achieved without distorting the trends of parameter sensitivity variation across the parameter range; otherwise, the normalization would influence the optimization. The various forms considered for the $\beta_{i,norm}/\alpha_{norm}$ normalization terms in Eq. (1) are outlined here, with their applicability to the optimization procedure detailed.

Hill (1992) normalizes the sensitivity by the input parameter resulting in a normalization term of $1/\beta_i$. In this case, the sensitivity term will remain dimensional. All terms in the quantitative metric have units of the output parameter and can therefore be summed. Although this method normalizes the sensitivities to have common units, they are not scaled by the input variable, and therefore a direct comparison is heavily weighted toward input parameters with larger magnitudes.

In the field of population biology, sensitivities are normalized by a multiplication of the input and output parameters, a technique known as elasticity (Link & Doherty, 2002). For application into the quantitative metric, the normalization term becomes α/β_i not only making the result dimensionless but also scaling such that comparisons are reasonable. However, as with the normalization only by the input parameter presented above, the scaling of the sensitivity by the input is not appropriate to this application of optimization. Alternatively, applying this normalization and making the sensitivity independent of the output parameter, α, will not influence the results of an optimization. The output factors all sensitivity functions equally, meaning that the combined minimum will occur at the same location, with only the value of the metric scaled by a function of the output.

To achieve a normalization of the input parameter that does not distort the sensitivity function yet still produces a dimensionless metric and scales the sensitivities to comparable magnitudes, the normalization of the input parameter must be constant. Possibilities include making the numerator of the normalization term in Eq. (2) equal to the minimum or maximum expected parameter ($\beta_{i,norm} = \beta_{i,min}$ or $\beta_{i,max}$), the range of expected parameters ($\beta_{i,norm} = \beta_{i,range}$), or the average expected parameter defined as the midpoint across an expected range ($\beta_{i,norm} = \beta_{i,avg}$). In this work, the average parameter $\beta_{i,avg}$ is considered favorable as it presents the least variation across the various parameter sensitivity functions. It has particular benefits when the minimum value of a parameter range does not approach zero, or the range is very narrow, for example, in the case of a friction angle. All these terms are considered subjective, and their values should be selected with consideration given to any relevant testing apparatus restrictions and general limits on the expected input parameters.

Finally, it can be seen that the proposed metric in Eq. (1) is obtained from the general form of Eq. (2) by selecting $\alpha_{norm} = \alpha$. Simplicity is the main factor motivating this choice, though one can envision other possibilities that are not explored here.

The final normalization, as presented in Eq. (1), combines normalization by the output parameter and average input parameter. This creates a quantitative metric that is independent of the output parameter and provides a consistent scaling across each of the independent sensitivity functions.

4 IMPLEMENTATION OF QUANTITATIVE METRIC

The quantitative metric has been designed with such versatility that it can be implemented for any problem. Here the implementation is demonstrated first through the extension of a spring and then with a uniaxial compression test. The quantitative metric is used to inform the decision of (1) a force- or displacement-controlled test and (2) the optimal configuration of a test considering all possible variables.

4.1 Spring extension

As described in Section 2, the first test considered in this paper involves the extension of a spring to determine the spring constant k. This example allows for the implementation of the proposed metric and comparison of tests in one of the simplest conceivable forms. The forward model for spring extension is

$$k = \frac{F}{\delta} \tag{3}$$

In an actual test, k is an unknown but fixed constant that is determined by applying displacement and measuring force (displacement control) or applying force and measuring displacement (force control). If k were considered constant in the optimization of Q, the force-displacement ratio for the spring extension test would also be constant due to the simplicity of the forward model. Therefore, optimization of the test is best analyzed by allowing all components of the test to be variable. In this case, that means that the k, F, and δ are considered variables in the optimization.

When a spring extension test is run as a displacement-controlled test, variables in Eq. (1) are $\alpha = F$, $\beta_1 = k$, and $\beta_2 = \delta$. It may be noted that sensitivity of the force to the displacement, which is controlled, is minimized to reduce the influence of potential variation in this quantity (i.e., error) on the measured output F. At the same time, the sensitivity of measured force F to the spring constant is maximized. The resulting quantitative metric is

$$Q_{spring,DC} = \frac{1}{2}\left[-\left(\frac{\partial F}{\partial k}\frac{k_{avg}}{F}\right)^2 + \left(\frac{\partial F}{\partial \delta}\frac{\delta_{avg}}{F}\right)^2\right] \quad (4)$$

Equation (4) includes the general form of the sensitivities $\partial\alpha/\partial\beta_i$. At first glance, one might be inclined to include the length of the spring, L, in the quantitative metric ($\beta_3 = L$). However, the inclusion of L results immediately in Eq. (4) since $\partial F/\partial L = 0$.

Upon substituting the derivatives of the forward model given by Eq. (3) into Eq. (4), the quantitative metric becomes

$$Q_{spring,DC} = \frac{1}{2}\left[-\left(\delta\frac{k_{avg}}{F}\right)^2 + \left(k\frac{\delta_{avg}}{F}\right)^2\right] \quad (5)$$

Further substitution using for the forward model for δ/F and k/F can be made to ascertain

$$Q_{spring,DC} = \frac{1}{2}\left[-\left(\frac{k_{avg}}{k}\right)^2 + \left(\frac{\delta_{avg}}{\delta}\right)^2\right] \quad (6)$$

Equation (6) reveals that the optimization can be analyzed independent of the output parameter F. Furthermore, Eq. (6) takes a sufficiently simple form such that the trend towards optimal can be seen without an optimization solver. In searching for the optimal test, the objective is to minimize Eq. (6). Without implementing bounds on the variables in Eq. (6), the analytical solution results in $\delta \to \infty$ and $k \to 0$. As k and δ must be positive constants within limits determined by practical considerations, constraints need to be added to bound the results. It can be seen by assessing Eq. (6) with constraints, Q will be minimized when δ is at its upper limit and k is at its lower limit.

For a practical application, k is a fixed property of the spring, and the practical outcome is that the optimal displacement-controlled test should have the largest displacement possible. The large displacement will minimize the influence of any variation in displacement on F and therefore produce the most accurate estimate of k.

For a force-controlled test, $\alpha = \delta$, $\beta_1 = k$ and $\beta_2 = F$. The sensitivity of the displacement to the force is minimized, and the sensitivity of the displacement to the spring constant is maximized:

$$Q_{spring,FC} = \frac{1}{2}\left[-\left(\frac{\partial \delta}{\partial k}\frac{k_{avg}}{\delta}\right)^2 + \left(\frac{\partial \delta}{\partial F}\frac{F_{avg}}{\delta}\right)^2\right] \quad (7)$$

Upon substituting the derivatives of the forward model given by Eq. (3), the quantitative metric becomes

$$Q_{spring,FC} = \frac{1}{2}\left[-\left(\frac{-F}{k^2}\frac{k_{avg}}{\delta}\right)^2 + \left(\frac{1}{k}\frac{F_{avg}}{\delta}\right)^2\right] \quad (8)$$

Further substitution using the forward model of Eq. (3) for F/δ and $1/\delta$ results in

$$Q_{spring,FC} = \frac{1}{2}\left[-\left(\frac{k_{avg}}{k}\right)^2 + \left(\frac{F_{avg}}{F}\right)^2\right] \quad (9)$$

The progression from Eq. (7) to Eq. (9) is similar to that for displacement control, where Eq. (9) includes the critical substitution revealing simplification of the optimization and independence from the output parameter δ.

The optimal configuration for a force-controlled test can be determined through the minimization of Q given by Eq. (9). As with displacement control, without the implementation of constraints, there is no sensible analytical solution. Once constraints are implemented, it can be seen that the ideal test will occur when F is maximized, and k is minimized. Similar to displacement control, for a practical application, when k is constant, a large force will minimize the sensitivity to variation in force, resulting in the optimal test. In both force- and displacement-controlled cases, a test of constant k can be improved by increasing the applied force and displacement.

For the purpose of this paper, force- and displacement-controlled tests will be compared directly. Justification for direct comparison of $Q_{spring,DC,}$ and $Q_{spring,FC}$ is not provided here due to constraints on the length of the article but will be provided in a forthcoming article by the co-authors. The comparison of Eqs. (6) and (9) can be used to conclude whether force or displacement control is preferred. A displacement-controlled test is preferred when

$$Q_{spring,DC} < Q_{spring,FC} \quad (10)$$

By substituting Eq. (6) and (9) into this expression, Eq. (10) becomes

$$\frac{1}{2}\left[-\left(\frac{k_{avg}}{k}\right)^2 + \left(\frac{\delta_{avg}}{\delta}\right)^2\right] < \frac{1}{2}\left[-\left(\frac{k_{avg}}{k}\right)^2 + \left(\frac{F_{avg}}{F}\right)^2\right] \quad (11)$$

With the aid of Eq. (3), simplification and rearrangement of Eq. (11) gives

$$k < \frac{F_{avg}}{\delta_{avg}} \quad (12)$$

Equation (12) shows that the decision to use force or displacement control depends on the comparison of the spring constant to the average force and displacement, F_{avg} and δ_{avg}, to which physical meaning can be ascribed as discussed in further detail below. This result is possible due to the normalization of sensitivities creating a quantitative metric that is independent of the output. The significant contribution of the substitution made to get from Eq. (5) to (6), and Eq. (8) to (9) results in the comparison of force- and displacement-controlled tests subsequently becoming only a comparison of the sensitivity of force to displacement and displacement to force.

Equation (12) suggests that in order to determine which test is preferable, one must first know critical details about the testing apparatus, referred to her simply as the "machine." Interpreting F_{avg} and δ_{avg} as representative of the force and displacement that the machine is capable of measuring or applying, the ratio F_{avg}/δ_{avg} can be interpreted as machine stiffness. Figure 2 illustrates how machine stiffness can be represented in the plane created by representing F_{avg} and δ_{avg} along the vertical and horizontal axes, respectively.

Because spring stiffness k is unknown, Eq. (12) also reveals that the optimal test cannot be determined without prior knowledge. Application of Eq. (12) to determine the optimal test requires an estimate of k. A machine capable of applying small displacement and measuring large force is optimal for a spring with large stiffness (k_{large}), as depicted in Figs. 2 and 3. Correspondingly, a machine capable of applying small force and measuring large displacement is optimal for springs with low stiffness (k_{small}). Significantly, Eq. (12) shows that as the spring constant decreases, the more likely it is that one should run a displacement-controlled test. Likewise, increasing spring stiffness makes it more likely that one should run a force-controlled test. Figure 3 summarizes this concept.

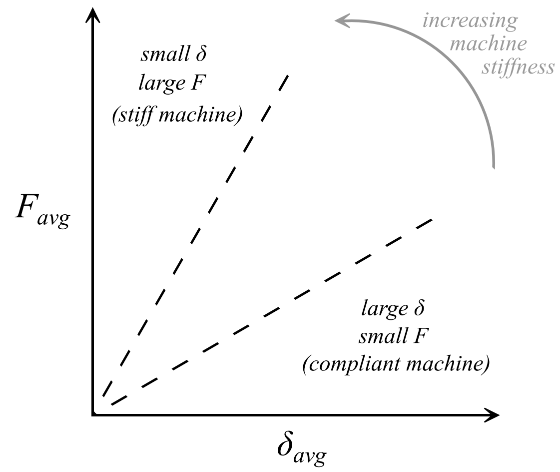

Figure 2. Varying machine stiffness represented in the place created by showing F_{avg} as a function of δ_{avg}

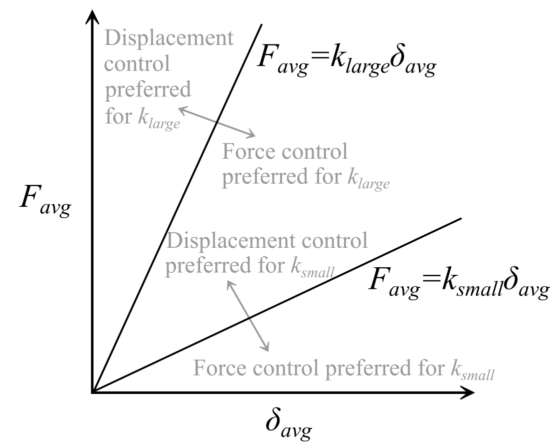

Figure 3. Normalized force controlled to displacement comparison for spring extension

For the springs shown schematically in Fig. 1, assuming they are the same material, the spring represented by stiffness k_1 likely has a lower k than that of k_4. It is, therefore, likelier that the spring with stiffness k_1 should be tested in displacement control. This is indicated in Figure 3, which shows that smaller values of k produce a greater area over which displacement control is preferred. For the fourth spring, with a higher k_4, it is likelier that force control will be optimal. The exact threshold determining which test is optimal depends on the machine stiffness F_{avg}/δ_{avg} and the exact value of k.

4.2 Uniaxial compression

The uniaxial compression test appears regularly in geotechnical engineering and includes additional geometric components compared to the previous example. The forward model for a uniaxial compression test is

$$E = \frac{FL}{\delta bd} \tag{13}$$

Variables in Eq. (13) are as follows: E = elastic modulus of the material, F = force, δ = displacement, L = sample length, b = sample width, and d = sample depth. While a circular cross-section is typically utilized in practice, a rectangular cross-section is here used to highlight the influence of geometric variables.

The added complexity of additional components allows the optimization to hold the material parameter (elastic modulus E) constant and optimize overall geometric components, as well as force and displacement. This is of benefit to a situation where the material for testing is known, and the determination of the ideal testing configuration is of interest.

For the displacement-controlled uniaxial compression test, $\alpha = F$, and $\beta = [E, \delta, L, b, d]$. The resulting quantitative metric is

$$Q_{uni,DC} = \frac{1}{2}\left[-\left(\frac{\partial F}{\partial E}\frac{E_{avg}}{F}\right)^2 + \left(\frac{\partial F}{\partial \delta}\frac{\delta_{avg}}{F}\right)^2 + \left(\frac{\partial F}{\partial L}\frac{L_{avg}}{F}\right)^2 + \left(\frac{\partial F}{\partial b}\frac{b_{avg}}{F}\right)^2 + \left(\frac{\partial F}{\partial d}\frac{d_{avg}}{F}\right)^2\right] \tag{14}$$

Equation (12) shows the general form including derivatives, and by making the substitution for the derivatives using the forward model of Eq. (13), the quantitative metric becomes

$$Q_{uni,DC} = \frac{1}{2}\left[-\left(\frac{\delta bd}{L}\frac{E_{avg}}{F}\right)^2 + \left(\frac{Ebd}{L}\frac{\delta_{avg}}{F}\right)^2 + \left(\frac{-E\delta bd}{L^2}\frac{L_{avg}}{F}\right)^2 + \left(\frac{E\delta d}{L}\frac{b_{avg}}{F}\right)^2 + \left(\frac{E\delta b}{L}\frac{d_{avg}}{F}\right)^2\right] \tag{15}$$

This can be simplified by making an additional substitution for the forward model (Eq. 13) into each of the sensitivity terms, resulting in

$$Q_{uni,DC} = \frac{1}{2}\left[-\left(\frac{E_{avg}}{E}\right)^2 + \left(\frac{\delta_{avg}}{\delta}\right)^2 + \left(\frac{L_{avg}}{L}\right)^2 + \left(\frac{b_{avg}}{b}\right)^2 + \left(\frac{d_{avg}}{d}\right)^2\right] \tag{16}$$

As was the case for spring extension, the result of Eq. (16) is independent of the output parameter F.

For the force-controlled uniaxial compression test, $\alpha = \delta$ and $\beta = [E, F, L, b, d]$. The resulting quantitative metric is

$$Q_{uni,FC} = \frac{1}{2}\left[-\left(\frac{\partial \delta}{\partial E}\frac{E_{avg}}{\delta}\right)^2 + \left(\frac{\partial \delta}{\partial F}\frac{F_{avg}}{\delta}\right)^2 + \left(\frac{\partial \delta}{\partial L}\frac{L_{avg}}{\delta}\right)^2 + \left(\frac{\partial \delta}{\partial b}\frac{b_{avg}}{\delta}\right)^2 + \left(\frac{\partial \delta}{\partial d}\frac{d_{avg}}{\delta}\right)^2\right] \tag{17}$$

Making the substitution for the derivatives using the forward model (Eq. 13) gives

$$Q_{uni,FC} = \frac{1}{2}\left[-\left(\frac{-FL}{E^2bd}\frac{E_{avg}}{\delta}\right)^2 + \left(\frac{L}{Ebd}\frac{F_{avg}}{\delta}\right)^2 + \left(\frac{F}{Ebd}\frac{L_{avg}}{\delta}\right)^2 + \left(\frac{-FL}{Eb^2d}\frac{b_{avg}}{\delta}\right)^2 + \left(\frac{-FL}{Ebd^2}\frac{d_{avg}}{\delta}\right)^2\right] \tag{18}$$

Further substitution of the forward model (Eq. 13) in each of the sensitivity terms results in

$$Q_{uni,FC} = \frac{1}{2}\left[-\left(\frac{E_{avg}}{E}\right)^2 + \left(\frac{F_{avg}}{F}\right)^2 + \left(\frac{L_{avg}}{L}\right)^2 + \left(\frac{b_{avg}}{b}\right)^2 + \left(\frac{d_{avg}}{d}\right)^2\right] \tag{19}$$

As in the case of the spring, by comparing the results for force and displacement control (Eqs. (16) and (19)), the following inequality indicating when displacement control test is preferred can be determined:

$$\frac{F}{\delta} < \frac{F_{avg}}{\delta_{avg}} \tag{20}$$

Upon substituting for F/δ, Eq. (20) can alternatively be written to show dependence on the material parameter and geometric configuration explicitly:

$$\frac{Ebd}{L} < \frac{F_{avg}}{\delta_{avg}} \quad (21)$$

Overall conclusions from this analysis of the uniaxial compression test are similar to those obtained for spring extension (Section 4.1). For the case of displacement control, Equation (16) for uniaxial compression takes a similar form to Eq. (6) for spring extension but with three additional terms corresponding to the three geometric parameters (L, b, and d). Equation (21) indicates that displacement control is preferred when the stiffness of the specimen, given by Ebd/L, is small relative to machine stiffness (F_{avg}/δ_{avg}).

Consideration to the optimal configuration of the uniaxial compression test for either force or displacement control is significantly more complicated due to the variability of five or six parameters in the optimization (depending on if the material parameter is variable or not), coupled with the nonlinear constraint introduced by Eq. (13). An optimization solver is required for such an analysis, which is considered in a forthcoming article by the co-authors.

5 CONCLUDING REMARKS

This paper proposes a quantitative metric that allows for optimization of testing configurations, including general cases extending well beyond the specific examples considered in this article.

The simple example of spring extension demonstrates how the proposed metric can be applied for the selection of the test type and ideal testing configuration. With the selected normalization method, the choice to run a force- or displacement-controlled test is effectively determined by machine stiffness, as well as spring stiffness. Since spring stiffness is unknown prior to testing, an estimate must be made for practical determination of the optimal configuration. The smaller the spring constant, the more likely a displacement-controlled test will be preferred. As the spring constant increases, so does the likelihood that force control is preferred. When assessing the force- and displacement-controlled tests individually, the quantitative metric decreases (becomes closer to optimal) as the spring stiffness is reduced. In other words, accuracy generally deteriorates as stiffness increases. For a displacement-controlled test, the objective is to minimize the sensitivity of the displacement with respect to the force, resulting in a large force. Alternatively, in a force-controlled test, the objective is to minimize the sensitivity of force with respect to displacement, resulting in a preferred large displacement.

Furthermore, implementation of the quantitative metric for the uniaxial compression test displays a similar discriminator with respect to the comparison of force and displacement control: the stiffness of the material. The added complexity of additional geometric components means that the selection of the test is not entirely dependent on the material parameter. However, as an area of future work, optimization can be completed numerically for specific values of the material parameter.

Future work will implement the quantitative metric to compare different testing methods with larger numbers of contributing factors and implement for use in finite element analysis for topology optimization for optimal soil parameter determination. Application of the proposed metric to more complex problems, including those relevant to laboratory testing and *in situ* testing in geotechnical engineering, is a potentially fruitful area of future research.

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