Undrained uplift capacity of deeply embedded strip anchors in non-uniform soil

S. B. Yu, J. P. Hambleton* and S. W. Sloan

ARC Centre of Excellence for Geotechnical Science and Engineering, The University of Newcastle, Callaghan, NSW, Australia

ABSTRACT: This paper presents new theoretical predictions of the undrained uplift capacity of deeply embedded inclined strip anchors in a soil stratum with a linear variation of strength with depth. Rigorous bounds on the theoretical uplift capacity are presented using upper and lower bound limit analysis. The effects of the strength gradient on the normalised uplift capacity and the predicted failure mechanisms are analysed. Overall, the effect of the strength gradient on the capacity is shown to be rather small (less than 16% for the range of strength gradients considered), while the influence of the gradient on the failure mechanism can be significant. It is shown that the coupled effects of anchor inclination and the strength gradient are well characterised by formulae that provide a simple means for refining uplift calculations in practical applications.

KEYWORDS: anchors; limit analysis

1. INTRODUCTION

Plate anchors are routinely used as foundations for resisting uplift and horizontal pullout forces. For example, they are often implemented in offshore applications to anchor platforms and floating structures to the seabed. Onshore, plate anchors are used to support guy wires for communication and transmission towers, and they are often placed behind retaining walls to act as tiebacks.

* Corresponding author. Address: ARC Centre of Excellence for Geotechnical Science and Engineering, University of Newcastle, University Drive, Callaghan, NSW 2308, Australia; Tel.: +61 2 4921 5893; Fax: +61 2 4921 6991 Email addresses: shengbingyu@outlook.com (S. B. Yu), James.Hambleton@Newcastle.edu.au (J. P. Hambleton), Scott.Sloan@Newcastle.edu.au (S. W. Sloan)
Short-term and long-term stability are considerations of paramount importance in the design of plate anchors. Accordingly, numerous methods for predicting the ultimate uplift (pullout) capacity with varying anchor types and soil conditions have been proposed. Most approaches are empirical and revolve around data obtained from laboratory model tests [1-9]. Theoretical solutions have been developed based on limit equilibrium methods [10-13], finite element methods [14-22], cavity expansion theory [23] and limit analysis [4, 22, 24-30]. Due to wide variations in the soil type, anchor shape, contact conditions, embedment depth, anchor inclination, and loading type, rigorous theoretical approaches for predicting uplift capacity have been slow to replace empirical methods.

The majority of previous studies assume uniform strength throughout the soil mass in which the anchor is embedded, but the profile of shear strength is often non-uniform. The need to develop solutions for a non-uniform strength profile is perhaps best reflected in the study by O'Neill et al. [22], who focus on the performance of drag anchors in clays. In this work and those to follow (e.g., [31-33]), pullout capacity is evaluated simply by averaging the soil strength in the region adjacent to the anchor and then relying on theoretical predictions obtained for a uniform strength profile. In the words of the original study [22], “this is clearly a simplification, and refinement will be needed in the future…to take account of variations of soil strength with depth in a more rigorous fashion.” While some previous studies analyze the case where strength increases linearly with depth [16,17,26], no analytical solutions for the uplift capacity of deeply embedded anchors have been proposed. The finite element analyses completed by Yu et al. [16] include deeply embedded vertical and horizontal anchors, but the coupled effects of anchor inclination and the strength gradient for a deep anchor at arbitrary inclination are not addressed.

This paper presents a rigorous analysis of the undrained uplift capacity of inclined, deeply embedded anchors in stratum of soil for which the shear strength varies linearly with depth. Within the framework of limit analysis [34], newly derived semi-analytical upper bound solutions are compared with numerical results obtained using a recently developed upper bound numerical approach known as the “block set mechanism” [35, 36] as well as lower and upper bound finite element limit analysis (FELA). Tacit assumptions are that the material is perfectly plastic and obeys the Tresca yield criterion. Plain strain is also assumed, and therefore attention is restricted to the case of thin rectangular strip anchors, for which the length greatly exceeds
the width. For simplicity, both the anchor and the soil are considered to be weightless, recognizing that the influence of soil weight is minimal for undrained analysis of deeply embedded anchors. The analysis provides verified equations for use in geotechnical design, and it lends insights into the coupled effects of anchor inclination and the strength gradient on the normalised uplift capacity and shape of the failure mechanism.

2. PROBLEM DEFINITION

Figure 1 shows a schematic of the problem analyzed in this study. The anchor’s width is denoted by \( B \), and its thickness is taken to be negligibly small. The width \( B \) is also assumed to be much smaller than the anchor’s depth of embedment, such that the collapse mechanism does not extend to the surface. The anchor is inclined at angle \( \alpha \) from the horizontal. At the two interfaces between the anchor and the soil, two types of contact are considered: perfectly smooth (frictionless) and perfectly rough (adhesion equal to the undrained shear strength). The direction of loading is taken to be perpendicular to the anchor, and the major unknown is the force per unit width on the anchor at collapse, i.e., the ultimate load, denoted by \( Q_u \). It is assumed that soil remains in contact with the anchor on both interfaces and that tensile stresses are permitted to develop (i.e., no breakaway). The undrained shear strength \( c_u \) is given by

\[ c_u = c_{u0} + \rho z \]  

(1)

where \( z \) is depth, \( \rho \) is the strength gradient (\( \rho > 0 \)), and \( c_{u0} \) is the undrained shear strength at the datum \( z = 0 \). For convenience, the anchor’s midpoint (centroid) is selected as the datum \( z = 0 \). The parameter \( c_{u0} \) can therefore be considered as the spatial average of the shear strength in the vicinity of the anchor, and thus the average shear strength defined by O’Neill et al. [22] and

![Fig. 1 Problem definition](image1.png)
Dimensional analysis reveals that the problem is characterized by the following dimensionless groups: \( \frac{Q_0}{c_{\text{u0}}}B \), \( \frac{\rho B}{c_{\text{u0}}} \), and \( \alpha \). The first of these is \( N_c \), the so-called capacity factor, and this factor is a heretofore unknown function of \( \frac{\rho B}{c_{\text{u0}}} \) and \( \alpha \):

\[
N_c = \frac{Q_0}{c_{\text{u0}}}B = f\left(\frac{\rho B}{c_{\text{u0}}}, \alpha\right)
\]

The dimensionless strength gradient, \( \frac{\rho B}{c_{\text{u0}}} \), is considered to vary between \( 0 \leq \frac{\rho B}{c_{\text{u0}}} \leq 0.3 \). This range is based on typical values (cf. [37, 38]) as well as the physical limitation that the strength cannot be less than zero in the vicinity of the anchor. The analysis focuses on \( 0 \leq \alpha \leq 90^\circ \), although the analysis can be readily extended to the full range \( 0 \leq \alpha \leq 360^\circ \), which includes scenarios involving failure in bearing rather than uplift.

3. LIMIT ANALYSIS

Three different limit analysis techniques are used to analyze the problem defined in Section 2: (1) a semi-analytical approach based on upper bound limit analysis, (2) the block set mechanism proposed by Yu et al. [35, 36], which is also based on upper bound limit analysis, and (3) lower and upper bound finite element limit analysis (FE LA). Upper bound limit analysis rests on consideration of kinematical velocity fields (collapse mechanisms), and it rigorously brackets the true ultimate load from above for loads inducing collapse. Lower bound limit analysis considers statically admissible stress fields as a means of bounding the true ultimate load from below. Additional details regarding limit analysis are described in the comprehensive monograph by Chen [34].

3.1. Semi-analytical Approach

Solutions for the unknown function \( f \) in Eq. (2) can be obtained within the framework of upper bound limit analysis by postulating the kinematically admissible collapse mechanisms shown in Fig. 2. In both cases, the anchor moves with velocity \( v_0 \) normal to the anchor surface. The first mechanism, shown in Fig. 2(a), is applicable to rough anchors, and it consists of two regions that move as rigid bodies together with anchor (\( OCE \) and \( OFE \)) and two zones of continuous shear bounded by circular arcs (\( OFC \) and \( ECF \)). The second mechanism, shown in Fig. 3(b), is valid for smooth anchors, and it involves two rigid regions that slide relative to the
Fig. 2 Upper bound mechanisms for a deeply embedded strip anchor: (a) perfectly rough and (b) perfectly smooth.

Fig. 3 Failure mechanism for a deeply embedded strip anchor in homogeneous undrained clay.

anchor ($OCE$ and $OFE$) and only one zone of continuous shear ($OFC$). Both mechanisms can be considered extensions of the one analyzed by Rowe [4], depicted in Fig. 3(a), to compute upper bounds on the uplift capacity in homogeneous soil. Rowe (1978) indicated that the failure mechanism for a rough anchor is also applicable to smooth strip anchors, and indeed Meyerhof [39] earlier showed using the method of characteristics that the capacity factor $N_c = 3\pi + 2$ is independent of the contact conditions. For homogenous soil, the same ultimate load is in fact obtained for each of the three different mechanisms shown in Fig. 3, although the mechanisms for the smooth anchor are valid only when relative sliding is permitted at the soil-anchor interfaces. For heterogeneous soil characterized by a strength gradient, each of the mechanisms depicted in Fig. 3 gives different values for the ultimate load. Moreover, more accurate (smaller) upper bounds are obtained for the mechanisms shown in Fig. 2, which permit variation in the shape through the angles $\xi$ and $\eta$. 

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The full derivation of the capacity factors computed based on the mechanisms shown in Fig. 2 is provided in Appendices A and B. Based on the mechanism shown in Fig. 2(a), the analytical upper bound solution for a rough anchor is given by

\[
N_c = \frac{\sin \eta \cos \xi}{\sin(\xi + \eta)} \left[ (2 - \frac{\rho B}{c_{u0}} \sin \alpha) (2\pi - 2\xi) - 3 \frac{\rho B \sin \alpha \sin \eta \sin \xi}{c_{u0} \sin(\xi + \eta)} \right] \\
+ \frac{\cos \eta \sin \xi}{\sin(\xi + \eta)} \left[ (2 + \frac{\rho B}{c_{u0}} \sin \alpha) (2\pi - 2\eta) + 3 \frac{\rho B \sin \alpha \sin \eta \sin \xi}{c_{u0} \sin(\xi + \eta)} \right] \\
+ \frac{\sin \eta \sin \xi}{\sin(\xi + \eta)} \left[ 4 - \frac{\rho B}{c_{u0}} \sin \alpha + 3 \frac{\rho B \sin \alpha \sin \eta \cos \xi}{c_{u0} \sin(\xi + \eta)} \right]
\]  

(3)

The solution for a smooth anchor based on the mechanism shown in Fig. 2(b) is

\[
N_c = \left(2 - \frac{\rho B}{c_{u0}} \sin \alpha \right) (\pi + 2\eta) + \frac{\cos \eta}{\sin \eta} - \frac{\rho B}{c_{u0}} \sin \alpha \sin(2\eta)
\]  

(4)

Since Eqs. (3) and (4) provide upper bounds on the true collapse load, the best estimates of \( N_c \) correspond to the values of the angles \( \xi \) and \( \eta \) that minimize the expressions. In this paper, the minimization process was carried out numerically, since analytical solutions could not be found.

For the rough anchor, the optimal value of \( \xi \) tends to decrease as \( \rho B/c_{u0} \) increases, while the optimal value of \( \eta \) tends to increase. For the smooth anchor, the optimal value of \( \eta \) increases only moderately as \( \rho B/c_{u0} \) increases. Over the range \( 0 \leq \alpha \leq 90^\circ \) and \( 0 \leq \rho B/c_{u0} \leq 0.3 \), the optimal values vary within the ranges \( 18^\circ < \xi < 45^\circ \) and \( 45^\circ < \eta < 72^\circ \) for the rough anchor and \( 45^\circ \leq \eta < 50^\circ \) for the smooth anchor.

While the analysis in this paper concentrates on uplift, the semi-analytical approach is in fact applicable for \( 0 \leq \alpha \leq 360^\circ \), where \( 90^\circ < \alpha < 180^\circ \) corresponds to bearing rather than uplift. It may be readily shown that \( N_c \) is symmetric about \( \alpha = 0, 90^\circ, 180^\circ, \) and \( 360^\circ \), and thus the results for \( 0 \leq \alpha \leq 90^\circ \) immediately extend to the full range \( 0 \leq \alpha \leq 360^\circ \).

### 3.2. Block Set Mechanism

To calculate upper bounds using the block set mechanism [35, 36], the collapse mechanism shown in Fig. 4 is assumed. The mechanism consists of sliding triangular rigid blocks separated by so-called velocity discontinuities [34,40], and it is similar to the one depicted in Fig. 2(a) for a rough anchor, whilst also being capable of representing the mechanism shown in Fig. 2(b).
Fig. 4 Assumed configuration of rigid blocks in the “block set mechanism” [35, 36] for (a) a horizontal strip anchor, where only half of the symmetric configuration is shown, and (b) inclined strip anchor.

for a smooth anchor by allowing the velocity in some blocks to be zero, as illustrated in Fig. 4(b). As an upper bound approach, the method rests on finding the geometry that minimizes the uplift capacity. This is achieved by treating the interior angles and edge lengths of the blocks as unknowns, and subsequently optimizing these unknowns using a hybrid genetic algorithm (GA) combined with a pattern search method (or direct search method). The method consists of performing an initial optimization for a specified number of blocks and then introducing additional blocks adaptively by interpolation, such that the final mechanism can include hundreds of blocks. Full details regarding the numerical technique are given by Yu [35].

3.3. Finite Element Limit Analysis (FELA)

Rigorous lower and upper bounds on the ultimate capacity are also obtained using the algorithms for FELA developed at The University of Newcastle, Australia [41-43] and summarized by Sloan [44]. These formulations have been used successfully to compute capacities for anchors in clay and sand (e.g., [26, 27, 29]), and readers are referred to the
original papers for further details. A major advantage of FELA over the semi-analytical approach and the block set method is that the mechanism is determined as part of the solution process rather than being specified beforehand, which in some instances allows for much more refined calculation of the bounds.

The calculations based on FELA were completed by regarding the infinitesimally thin plate anchor as a line segment (i.e., discontinuity) with ‘positive’ and ‘negative’ faces corresponding to the front and back soil-anchor interfaces. The upper bound calculations were performed by constraining the normal component of velocity to be equal to the anchor velocity $v_0$ on both interfaces, thus modelling uplift without breakaway, and by taking the dissipation per unit length at the interfaces to be either zero (smooth anchor) or the product $c_{uz}[v]$ (rough anchor), where $[v]$ denotes the relative tangential velocity at the interface (see [34,44] for further details). In the lower bound calculations, the shear stress at the interfaces was either zero (smooth anchor) or potentially non-zero with the constraint $|\tau| \leq c_{uz}$ (rough anchor), where $\tau$ is the shear stress at the interface. The line segment representing the plate anchor was centered within a square domain of soil with a depth and breadth of approximately $8B$. Along the boundaries, so-called extension elements [44] with shear strength $c_{uz}$ calculated according to Eq. (1) were implemented for lower bound limit analysis, and the velocities were taken to be zero for upper bound limit analysis. Observing that a physically meaningful solution requires $c_{uz} \geq 0$ everywhere in the domain, the maximum strength gradient that could be considered varied from $\rho B/c_{u0} = 0.2$ to 0.3 given the size of the domain selected, and hence the results plotted in Section 4 do not always span the full range $0 \leq \rho B/c_{u0} \leq 0.3$ considered to cover the full spectrum of values encountered in practice [37, 38].

4. RESULTS AND DISCUSSION

4.1. Ultimate capacity

Figures 5 and 6 compare values of the capacity factor $N_c$ evaluated using the three techniques described in Section 3 for the rough and smooth anchors, respectively. As a verification of the results, it can be seen that (1) the semi-analytical upper bounds are very close to the values obtained using the block set mechanism for all values of $\alpha$ and $\rho B/c_{u0}$ and (2) the upper bounds calculated from the semi-analytical approach and the block set method are always greater than lower bounds from FELA. In most instances the upper bounds obtained from the semi-
analytical approach and the block set method are less than the upper bounds evaluated using FELA and therefore closer to the true collapse load. The exception is for inclined rough anchors (Figures 5(b) and 5(c)) with relatively large values of the dimensionless gradient $\rho B/c_{u0}$. In these cases, the values obtained using FELA are the least of the upper bounds, suggesting that the alternative methods may oversimplify the collapse mechanism. The nature of this oversimplification is discussed further in Section 4.2.

Figures 5 and 6 show that the capacity of a deep strip anchor in all cases decreases as the strength gradient increases, and the reduction is more significant for a smooth anchor than for a rough anchor. For a horizontal anchor ($\alpha = 0$), the reduction is minimal, i.e., less than 1%. For the vertical anchor ($\alpha = 90^\circ$), the reduction is more significant but still relatively small, with less than 16% variation in $N_c$ over the range $0 \leq \rho B/c_{u0} \leq 0.3$. The decrease in capacity with increasing inclination is consistent with the conclusion reached by Yu et al. [16].

As shown in Figures 5 and 6, the values of the capacity factor $N_c$ obtained using the semi-analytical approach and the block set mechanism are well approximated by the following equations:

\begin{align*}
N_c &= N_{c0} \left[ 1 - \left( \frac{\rho B}{c_{u0}} \sin \alpha \right)^2 \right] - \frac{\pi}{4} \left( \frac{\rho B}{c_{u0}} \right)^2 \quad (5) \\
N_c &= N_{c0} \left[ 1 - \frac{\rho B}{2c_{u0}} \sin \alpha - \frac{1}{4\pi} \left( \frac{\rho B}{c_{u0}} \right)^2 \right] \quad (6)
\end{align*}

Equations (5) and (6) are for rough and smooth anchors, respectively, and $N_{c0} = 3\pi + 2$ is the capacity factor for homogenous soil. These equations provide a means for directly evaluating $N_c$, without the need for minimization as required for Eqs. (3) and (4).
Fig. 5 Capacity factors for a rough anchor: (a) $\alpha = 0$, (b) $\alpha = 45^\circ$, and (c) $\alpha = 90^\circ$.  

(a) 

(b) 

(c)
Fig. 6 Capacity factors for a smooth anchor: (a) $\alpha = 0$, (b) $\alpha = 45^\circ$, and (c) $\alpha = 90^\circ$. 
### 4.2. Collapse Mechanisms

Figures 7-10 compare the collapse mechanisms obtained using the block set mechanism and FELA for varying inclination angles $\alpha$ and dimension strength gradients $\rho B/c_{u0}$. Figures 7 and 8 pertain to a rough anchor, and Figures 9 and 10 are for a smooth anchor. The mechanisms predicted using the semi-analytical approach were found to be virtually identical to those ascertained using the block set mechanism, and therefore the former are not shown in the figures.

For a horizontal anchor characterized by $\alpha = 0$, the configuration is symmetrical about the vertical line passing through the anchor’s midpoint, and only half of the geometry is considered (Fig. 7(a)). With $\alpha = 0$, it can be seen that the strength gradient has a limited impact on the collapse mechanism, and this agrees with trends in the predicted capacity (Figs. 5(a) and 6(a)). Similar results were obtained for the smooth anchor, and therefore the collapse mechanisms corresponding to $\alpha = 0$ are not shown in Figs. 8-9.

For inclinations of $\alpha = 45^\circ$ and $90^\circ$, the effects of the strength gradient are more pronounced than for a horizontal anchor. In particular, the strength gradient causes the mechanism to skew towards weaker material. For an inclined rough anchor (Figs. 7 and 8), a large strength gradient causes the mechanism to become practically one-sided (e.g., $\rho B/c_{u0} = 0.2$ in Figs. 8(a) and 8(b)), and for an inclined smooth anchor (Figs. 9 and 10), the results from FELA confirm that the mechanism is always one-sided, as assumed in the semi-analytical approach.

Overall, the collapse mechanisms predicted based on the semi-analytical approach, block set method, and FELA are in good agreement. Close inspection of Figs. 8(a) and (8b) reveals that the mechanisms predicted using FELA differ from those assumed in the semi-analytical approach and the block set mechanism for a rough anchor. Namely, the mechanism consists of no relative sliding on the leading interface but relative sliding on the trailing interface when the strength gradient becomes large (Figs. 5(b) and 5(c)). The mechanisms postulated in the semi-analytical approach and the block set mechanism (Figs. 2 and 4, respectively) do not permit such a motion, which evidently provides an improved estimate of the capacity factor $N_c$ for large inclination angles and strength gradients.
Fig. 7 Outline of optimized collapse mechanism evaluated using the block set method for a rough anchor: (a) $\alpha = 0$, (b) $\alpha = 45^\circ$, and (c) $\alpha = 90^\circ$. Coordinates $x$ and $z$ are normalized by the footing width $B$. 
Fig. 8 Collapse mechanisms evaluated using FELA for a rough anchor: (a) $\alpha = 45^\circ$ and (b) $\alpha = 90^\circ$. Vectors show the nodal velocities, and contours indicate the rate of dissipation, from low (blue) to high (red).
Fig. 9 Outline of optimized collapse mechanism evaluated using the block set method for a smooth anchor: (a) $\alpha = 45^\circ$ and (c) $\alpha = 90^\circ$. Coordinates $x$ and $z$ are normalized by the footing width $B$. 
Fig. 10 Collapse mechanisms evaluated using FELA for a smooth anchor: (a) $\alpha = 45^\circ$ and (b) $\alpha = 90^\circ$. Vectors show the nodal velocities, and contours indicate the rate of dissipation, from low (blue) to high (red).
5. CONCLUSIONS

The undrained uplift capacity of inclined, deeply embedded strip anchors in soils with a linear variation of strength with depth has been analyzed using three different limit analysis techniques: a semi-analytical upper bound approach, an numerical upper bound method known as the “block set mechanism”, and upper and lower bound finite element limit analysis (FELA). Each of the three techniques provides rigorous bounds on the true collapse load. The upper bounds evaluated based on the semi-analytical approach and block set mechanism are found to be very similar for all inclination angles $\alpha$ and normalized strength gradients $\rho B/c_{u0}$, these upper bounds are in most instances lower (better) than the upper bounds from FELA. However, for inclined rough anchors with a large strength gradient (e.g., $\alpha > 45^\circ$ and $\rho B/c_{u0} > 0.15$), FELA produces the least of the upper bounds due to intricacies in the failure mechanism that could not be captured with the simplified approaches. Examination of the failure mechanisms confirms that, as expected, a strength gradient tends to skew the collapse mechanism in the direction of weaker soil.

This study is the first of its kind to conclusively demonstrate the effect of the strength gradient on the capacity of deeply embedded strip anchors, for which the uplift capacity is typically evaluated using solutions for homogenous material and some measure of the average strength in the vicinity of the anchor (e.g., $c_{u0}$). The analysis shows that the strength gradient reduces the capacity of the anchor somewhat, and the effect is more pronounced when the anchor is inclined. Furthermore, the effect is more significant for smooth anchors than for rough anchors. However, the analysis indicates that the strength gradient reduces the capacity by less than 16% over the full range of strength gradients considered ($0 \leq \rho B/c_{u0} \leq 0.3$). The paper thus demonstrates the validity of strength averaging but provides useful formulae (Eqs. (5) and (6)) for calculating more exact values of capacity in instances where greater precision is desired.

Recognizing that the present study assumes deep embedment, and that the soil is assumed to remain in contact with the anchor on both the front and back interfaces (i.e., no breakaway), future studies may consider the effects of shallow embedment and alternative contact conditions, as well as the influence of soil stiffness and large deformations (cf. [18]). While these factors will reduce the anchor’s capacity in many instances, the changes in capacity owing to the presence of a strength gradient, as surmised relative to the capacity with a uniform strength profile, are expected to be similar to the variations observed in this study.
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APPENDIX A: RATE OF DISSIPATION IN REGIONS WITH VARYING SHEAR STRENGTH

Conventional methods for evaluating upper bounds analytically, which rest on equating the rate of dissipation due to plastic deformation to the rate of work done by external forces, typically consider only the case of constant (homogeneous) soil strength (e.g., Chen [34]). This Appendix presents a generalized technique for computing dissipation rates for collapse mechanisms in regions with an arbitrary variation of soil strength. The results are subsequently used in Appendix B to determine upper bounds using the semi-analytical approach (Section 3.1).

Choosing a polar coordinate system, the velocities in the radial and circumferential directions are denoted by \( u \) and \( v \), respectively. The radial and circumferential normal strain rates, denoted by \( \dot{\varepsilon}_r \) and \( \dot{\varepsilon}_\theta \), respectively, are defined as

\[
\dot{\varepsilon}_r = \frac{\partial u}{\partial v} \quad (A1)
\]

\[
\dot{\varepsilon}_\theta = \frac{\partial v}{r \partial \theta} - \frac{u}{r} \quad (A2)
\]

The shear strain rate \( \dot{\gamma}_r\theta \) is

\[
\dot{\gamma}_r\theta = \frac{v}{r} - \frac{\partial v}{\partial r} - \frac{\partial u}{r \partial \theta} \quad (A3)
\]

The dissipation rate per unit area \( \dot{d} \) is

\[
\dot{d} = c_u(\theta, r)|\dot{\gamma}_{\text{max}}| \quad (A4)
\]
where \( c_d(\theta,r) \) is the pointwise variable strength and \( \dot{\gamma}_{\text{max}} \) is the maximum engineering shear strain. The total dissipation rate \( \dot{D} \) for a region specified by the limits \( R_1, R_2, \theta_1 \) and \( \theta_2 \) is given by

\[
\dot{D} = \int_{R_1}^{R_2} \int_{\theta_1}^{\theta_2} d(\text{strain rate}) \, r \, dr \, d\theta
\]  
(A5)

For a radial zone of continuous shear, \( u = 0 \) and \( \dot{\gamma}_{\text{max}} = \dot{\gamma}_{\text{rad}} = v/r \), and the dissipation rate within the zone is given by

\[
\dot{D} = \int_{R_1}^{R_2} \int_{\theta_1}^{\theta_2} c_d(\theta,r)v(\theta,r) \, dr \, d\theta
\]  
(A6)

Along velocity discontinuities represented by circular arcs (e.g., arc CF in Fig. 2(a)), the dissipation is given by

\[
\dot{D} = R[v]\int_{\theta_1}^{\theta_2} c_u(\theta,r) \, d\theta
\]  
(A7)

where \([v]\) is the jump in the tangential component of velocity and \( R \) is the radius of the arc. Hence, the total dissipation including the dissipation rate within the zone and its outer circular arc is given by

\[
\dot{D} = \int_{R_1}^{R_2} \int_{\theta_1}^{\theta_2} c_d(\theta,r)v(\theta,r) \, dr \, d\theta + R[v]\int_{\theta_1}^{\theta_2} c_u(\theta,r) \, d\theta
\]  
(A8)

Along velocity discontinuities represented by straight line segments (e.g., line OC in Fig. 2(a)), the total dissipation is given by

\[
\dot{D} = [v]\int_{\theta_1}^{\theta_2} c_u(\Theta,r) \, dr
\]  
(A9)

where \( \Theta \) is the angle of the straight line in the polar coordinate system.

**APPENDIX B: DERIVATIONS FOR SEMI-ANALYTICAL APPROACH**

**Rough anchor**

The collapse mechanism for a rough anchor is divided into four parts (Fig. 2(a)): two radial shear zones (OCF and ECF) and two rigid blocks (OCE and OEF). The circumferential velocities within the radial shear zones OCF and ECF, denoted by \( v_1 \) and \( v_2 \) respectively, are related to the velocity of the anchor \( v_0 \) as follows

\[
v_1 = v_0 \cos \hat{\xi}
\]  
(B1)

\[
v_2 = v_0 \cos \eta
\]  
(B2)
where angles $\xi$ and $\eta$ are as shown in Fig. 2(a). The jump in the tangential component of velocity along lines $OC$ and $OF$, denoted by $[v_{01}]$, is given by

$$[v_{01}] = v_0 \sin \xi$$

(B3)

The velocity jump along lines $CE$ and $EF$, denoted by $[v_{02}]$, is given by

$$[v_{02}] = v_0 \sin \eta$$

(B4)

Since no relative sliding is permitted at the soil-anchor interfaces, the total rate of dissipation due to plastic deformation consists of the dissipation within the radial shear zones $OCF$ and $EFC$ (including the dissipation along the velocity discontinuities that bound these regions), denoted by $\dot{D}_{OCF}$ and $\dot{D}_{EFC}$ respectively, and the dissipation along the velocity discontinuities $OC$, $EC$, $OF$, and $EF$, denoted by $\dot{D}_{OC}$, $\dot{D}_{EC}$, $\dot{D}_{OF}$, and $\dot{D}_{EF}$, respectively. The total dissipation $\dot{D}$ is thus

$$\dot{D} = \dot{D}_{OCF} + \dot{D}_{OC} + \dot{D}_{OF} + \dot{D}_{EFC} + \dot{D}_{EC} + \dot{D}_{EF}$$

(B5)

The dissipation rates $\dot{D}_{OCF}$, $\dot{D}_{OC}$, and $\dot{D}_{OF}$ can be readily computed using the equations developed in Appendix A by considering point $O$ as the pole. In this case, the shear strength of the soil is given by

$$c_u(\theta, r) = c_{u0} - \frac{B\rho}{2} \sin \alpha - \rho r \sin \theta$$

(B6)

The dissipation rate $\dot{D}_{OCF}$ can be evaluated using Eq. (A8) with $R_1 = 0$, $R_2 = \overline{OC} = B \sin \eta / \sin(\xi + \eta)$, $\theta_1 = -\alpha + \xi$, and $\theta_2 = 2\pi - \alpha - \xi$:

$$\dot{D}_{OCF} = \frac{Bv_0 \sin \eta \cos \xi}{\sin(\xi + \eta)} \left[ (2c_{u0} - \rho B \sin \alpha)(2\pi - 2\xi) - 3\rho B \frac{\sin \alpha \sin \eta \sin \xi}{\sin(\xi + \eta)} \right]$$

(B7)

The dissipation rates $\dot{D}_{OC}$ and $\dot{D}_{OF}$ are computed with the aid of Eq. (A9), taking $[v] = v_{01}$ and $\Theta = -\alpha + \xi$ (for $OC$) or $\Theta = 2\pi - \alpha - \xi$ (for $OF$):

$$\dot{D}_{OC} + \dot{D}_{OF} = \frac{Bv_0 \sin \eta \sin \xi}{2 \sin(\xi + \eta)} \left[ (2c_{u0} - \rho B \sin \alpha) + \rho B \frac{\sin \alpha \sin \eta \cos \xi}{\sin(\xi + \eta)} \right]$$

(B8)

Upon considering point $E$ as the pole, one can compute the rates of dissipation $\dot{D}_{EFC}$, $\dot{D}_{EC}$, and $\dot{D}_{EF}$. The shear strength in this instance can be expressed as

$$c_u(\theta, r) = c_{u0} + \frac{B\rho}{2} \sin \alpha - \rho r \sin \theta$$

(B9)
The dissipation rate $\dot{D}_{EFC}$ is evaluated using Eq. (A8) with $R_1 = 0$,

$$R_2 = \overline{EC} = B \sin \xi / \sin (\xi + \eta), \ \theta_1 = \pi - \alpha + \eta, \text{ and } \theta_2 = 3\pi - \alpha - \eta:\n$$

$$\dot{D}_{EFC} = \frac{Bv_0 \cos \eta \sin \xi}{\sin (\xi + \eta)} \left[ \frac{(2c_{o0} + \rho B \sin \alpha)(2\pi - 2\eta) + 3\rho B \sin \alpha \sin \eta \sin \xi}{\sin (\xi + \eta)} \right]$$

(B10)

The dissipation rates $\dot{D}_{EC}$ and $\dot{D}_{EF}$ are given by Eq. (A9), taking $[v] = v_{02}$, and either $\Theta = \pi - \alpha + \eta$ (for $EC$) or $\Theta = 3\pi - \alpha - \eta$ (for $EF$):

$$\dot{D}_{EC} + \dot{D}_{EF} = \frac{Bv_0 \sin \eta \sin \xi}{2 \sin (\xi + \eta)} \left[ \frac{2c_{o0} + 2\rho B \sin \alpha \sin \eta \cos \xi}{\sin (\xi + \eta)} \right]$$

(B11)

Finally, the capacity factor $N_c = Q_u / Bc_{d0}$ is assessed by equating the total rate of dissipation $\dot{D}$, given by Eqs. (B5), (B7), (B8), (B10), and (B11), to the rate of work done by external forces $\dot{W} = Q_u v_0$. Upon equating $\dot{W} = \dot{D}$ and performing some manipulation, one arrives at Eq. (3) in the body of text.

**Smooth anchor**

The collapse mechanism for a smooth anchor is divided into three parts (Fig. 2(b)): one radial shear zone ($OCF$) and two rigid blocks ($OCE$ and $OEF$). The velocity within the radial shear zone, denoted by $v_1$, is identical to the velocities of the two rigid blocks and related to the velocity of the anchor $v_0$ as follows:

$$v_1 = \frac{v_0}{\sin \eta}$$

(B12)

While relative sliding occurs at the two soil-anchor interfaces, the assumption of a smooth interface implies zero dissipation, and the total rate of dissipation thus consists of the dissipation within the radial shear zone $OCF$ (including dissipation along the bounding velocity discontinuity) and along the velocity discontinuities $EC$ and $EF$. Denoting these rates of dissipation by $\dot{D}_{OCF}$, $\dot{D}_{EC}$, and $\dot{D}_{EF}$, the total rate of dissipation $\dot{D}$ is

$$\dot{D} = \dot{D}_{OCF} + \dot{D}_{EC} + \dot{D}_{EF}$$

(B13)

The dissipation rate $\dot{D}_{OCF}$ can be evaluated by considering point $O$ as the pole, such that the soil shear strength is

$$c_s(\theta, r) = c_{o0} - \sin \alpha \frac{B}{2} \rho - \rho r \sin \theta$$

(B14)
With the aid of Eq. (A8), taking \( R_1 = 0 \), \( R_2 = OC = B \sin \eta \), \( \theta_1 = \pi/2 - \alpha - \eta \), \( \theta_2 = 3\pi/2 - \alpha + \eta \), and \( \nu(\theta, r) = \nu_0/\sin \eta \), one finds

\[
\dot{D}_{OCF} = B\nu_0(2c_{u0} - \rho B \sin \alpha)(\pi + 2\eta) - \frac{3}{2} \rho B^2 \nu_0 \sin \alpha \sin(2\eta) \quad \text{(B15)}
\]

The dissipation rates \( \dot{D}_{CE} \) and \( \dot{D}_{EF} \) can be assessed by taking point \( E \) as the pole, in which case the soil shear strength is

\[
c_s(\theta, r) = c_{u0} + \sin \frac{B}{2} \rho - \rho r \sin \theta \quad \text{(B15)}
\]

Using Eq. (A9) with \( \nu_1 = \nu_0/\sin \eta \), \( R_1 = 0 \), \( R_2 = CE = EF = B \cos \eta \), and either \( \Theta = \pi - \alpha - \eta \) (for \( CE \)) or \( \Theta = \pi - \alpha + \eta \) (for \( EF \)), one finds

\[
\dot{D}_{CE} + \dot{D}_{EF} = B\nu_0(2c_{u0} + \rho B \sin \alpha) \cos \frac{\eta}{\sin \eta} - \rho B^2 \nu_0 \sin \alpha \cos \frac{3\eta}{\sin \eta} \quad \text{(B16)}
\]

Upon equating the rate of work done by external forces, \( \dot{W} = Q \nu_0 \), to the total rate of dissipation \( \dot{D} \) given by Eqs. (B13), (B15), and (B16), one arrives at Eq. (4) in the body of the text.

**REFERENCES**


